A Li-Yau and Aronson-Bénilan approach to study Keller-Segel

A. Fernández-Jiménez

Mathematical Institute
University of Oxford

Joint work with C. Elbar & F. Santambrogio

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Dictostelium discoideum

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Aggregation of *Dictostelium discoideum* to create a more complex structure. Reference *Wikipedia*.

Keller-Segel, [KS70; Pat53]

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$$\begin{cases} \partial_t \rho = \Delta \rho^m - \operatorname{div}(\rho \nabla u) & \text{in } (0,\infty) \times \mathbb{R}^d, \\ -\Delta u = \rho & \text{in } (0,\infty) \times \mathbb{R}^d, \end{cases} \tag{KS}$$

where $m = 2 - \frac{2}{d}$.

Goals:

- Dimension $d = 2$. A Li-Yau type estimate.
- Dimension $d \geq 3$. An Aronson-Bénilan type estimate.

Asymptotic behaviour. $d = 2$

Mass is preserved. $M := \int_{\mathbb{R}^2} \rho_0(x)\, dx = \int_{\mathbb{R}^2} \rho(x)\, dx$.

Second moment is **lineal** [BDP06].
$\frac{d}{dt} M_2(t) = \frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \rho(x)\, dx = 4M \left(1 - \frac{M}{8\pi}\right)$.

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$$\rho(t) \to M\delta_0 + f \quad \text{as } t \nearrow \infty.$$
- **Supercritical**. $M > 8\pi$, [HV96; RS14; CGMN22]
$$\rho(t) \to M\delta_0 + f \quad \text{as } t \nearrow T < \infty.$$

The so-called **chemotactic collapse**.

Using a HLS inequality on the energy we find a critical mass M_c.

➢ **Subcritical**. $M < M_c$, diffusion [Bed11; BCL09].

➢ **Critical**. $M = M_c$, globally well-posed and L^∞ norm globally bounded in time [BCL09].

➢ **Supercritical**. $M > M_c$, all radial solutions blow up in finite time [BK13].

Li-Yau and Aronson-Bénilan

➢ **Li-Yau** [LY86]. The heat equation $\partial_t \rho = \Delta\rho = \mathrm{div}(\rho\nabla(\log\rho))$,

$$\Delta(\log\rho(t,\cdot)) \geq -\frac{d}{2t}.$$

➢ **Aronson-Bénilan** [AB79]. PME $\partial_t \rho = \Delta\rho^m = \mathrm{div}(\rho\nabla(\frac{m}{m-1}\rho^{m-1}))$,

$$\Delta\left(\frac{m}{m-1}\rho^{m-1}(t,\cdot)\right) \geq -\frac{\alpha}{t}, \qquad \alpha = \frac{d}{d(m-1)+2}.$$

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✍ Li-Yau and Aronson-Bénilan for **non-local aggregation**-diffusion???

➟ L^∞ bounds on ρ in terms of $\delta := \inf_{x\in\mathbb{R}^d}\Delta v(x)$.

1. Differential inequality on the pressure.
2. L^∞ estimates.
3. Li-Yau and Aronson-Bénilan estimates.
4. Discussions and new implications.

Overview

1. Differential inequality on the pressure.
2. L^∞ estimates.
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The pressure

$$v(t,x) = \begin{cases} \log \rho(t,x) - u(t,x), & d = 2, \\ \dfrac{m}{m-1}\,\rho^{m-1}(t,x) - u(t,x), & d \geq 3, \quad \left(m = 2 - \dfrac{2}{d}\right). \end{cases}$$

Let $x_0 = x_0(t)$ be a point achieving the minimum of $\Delta v(t,\cdot)$

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☞ **Differential inequality**.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta v(t,x_0) \geq (\Delta v(t,x_0))^2 - \frac{(d-1)^2}{2d}\sup_{i,j}\|\partial_{ii}u - \partial_{jj}u\|_{L^\infty}^2$$

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☞ **Estimate** $\|\partial_{ii}u - \partial_{jj}u\|_{L^\infty}$.

$$\sup_{i,j}\|\partial_{ii}u - \partial_{jj}u\|_{L^\infty} \lesssim |\widetilde{\delta}|^{d/(d+2)}\|\rho\|_{L^2}^{4/(d+2)},$$

where $\delta = \min(\Delta v)$ and $\widetilde{\delta} = \max(\delta, \|\rho\|_{L^\infty})$

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L^∞ estimates. Dimension 2

Let us remember that

$$v(x) = \log \rho(x) - u(x) \quad \text{and} \quad \Delta v(x) = \Delta \log \rho(x) + \rho(x).$$

We want to study L^∞ estimates in terms of

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- **Small mass**. Assume $0 < M < \frac{8\pi}{e}$.
- **Subcritical mass**. Assume $0 < M < 8\pi$.
- **Critical mass**. Assume $M = 8\pi$.

Small mass

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☞ Arbitrary x_0 and $r > 0$

$$\log \rho(x_0) \leq \fint_{B_r(x_0)} \log \rho + \frac{|\widetilde{\delta}|}{8} r^2 \qquad \text{[Subharmonic]}$$

$$\leq \log \left(\fint_{B_r(x_0)} \rho \right) + \frac{|\widetilde{\delta}|}{8} r^2 \leq \log \left(\frac{M}{\pi r^2} \right) + \frac{|\widetilde{\delta}|}{8} r^2 \qquad \text{[Jensen]}$$

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- **Optimal value** $r = \sqrt{8/|\widetilde{\delta}|}$.

$$\log \rho(x_0) \leq \log(M|\widetilde{\delta}|/8\pi) + 1 \qquad \Rightarrow \qquad \rho(x_0) \leq (Me/8\pi)|\widetilde{\delta}|.$$

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☞ L^∞ **bound.** $\Delta v \geq \delta \Rightarrow \Delta \log \rho \geq \delta - \rho$

$$\|\rho\|_{L^\infty} \leq \frac{Me}{8\pi}|\delta - \|\rho\|_{L^\infty}| \qquad \Rightarrow \qquad \|\rho\|_{L^\infty} \leq \frac{Me}{8\pi - Me}|\delta|.$$

Subcritical mass

Assume $0 < M < 8\pi$ and $\delta \neq 0$. Let us argue by **contradiction**.

☞ **Sequence**. $\Delta \log \rho_n + \rho_n \geq \delta$ and $\|\rho_n\|_{L^\infty} \geq n\delta$.

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☞ **Scaling**. Consider

$$\eta_n(x) = \lambda_n^2 \rho_n(\lambda_n x) \quad \text{where} \quad \lambda_n^2 = \frac{1}{\|\rho_n\|_{L^\infty}}$$

and we notice

$$\int_{\mathbb{R}^2} \eta_n = M \quad \text{and} \quad \|\eta_n\|_{L^\infty} = 1.$$

Furthermore, $\Delta \log \eta_n + \eta_n = \Delta \log \rho_n(\lambda_n \cdot) + \lambda_n^2 \rho_n(\lambda_n \cdot) \geq \delta \lambda_n^2$.

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☞ **Compactness**. The sequence η_n converges to η such that

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Subsolution of the Liouville equation with mass $\leq M < 8\pi$.

Lemma (Subsolution of the Liouville equation)

Suppose $h\colon \mathbb{R}^2 \to [-\infty, \infty)$ *is a function in* $L^\infty(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$ *such that*

$$\begin{cases} \Delta h + e^h \geq 0 & \text{in the set } \{ (x,y) : h(x,y) > t \} \text{ for all } t \in \mathbb{R}, \\ \displaystyle\int_{\mathbb{R}^2} e^h < +\infty. \end{cases}$$

Then $\int_{\mathbb{R}^2} e^h \, \mathrm{d}x \geq 8\pi$.

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Then $\int_{\mathbb{R}^2} e^h \, \mathrm{d}x \geq 8\pi$.

☞ **Contradiction**. Take $\eta = e^h$. Since $M < 8\pi$ contradiction.

☞ Due to the scaling $\|\rho\|_{L^\infty} \leq C|\delta|$.

Further cases

- **Dimension 2, $M = 8\pi$.** Contradiction + tightness

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Example: dimension 3. Multiply by h, integrate and integrate by parts

$$\|\nabla h\|_{L^2}^2 \leq \frac{m-1}{m}\|h\|_{L^4}^4.$$

Gagliardo-Nirenberg. $\|h\|_{L^4}^4 \leq C_{GN}\|\nabla h\|_{L^2}^2\|h\|_{L^3}^2 \leq C_{GN}\frac{m-1}{m}\|h\|_{L^3}^2\|h\|_{L^4}^4.$

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- Critical mass? Analogous to dimension 2.

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Li-Yau and Aronson-Bénilan

Recall $$\frac{\mathrm{d}}{\mathrm{d}t}\Delta v(t,x_0) \geq (\Delta v(t,x_0))^2 - \frac{(d-1)^2}{2d}\sup_{i,j}\|\partial_{ii}u - \partial_{jj}u\|^2_{L^\infty}$$

$\sup_{i,j}\|\partial_{ii}u - \partial_{jj}u\|_{L^\infty} \lesssim |\widetilde{\delta}|^{d/(d+2)}\|\rho\|_{L^2}^{4/(d+2)}$ and $\widetilde{\delta} = \max(\delta, \|\rho\|_{L^\infty})$.

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Furthermore,
$$\|\rho(t,\cdot)\|_{L^\infty} \leq \frac{C}{t} \quad \text{for a.e. } (t,x) \in (0,T)\times\mathbb{R}^d$$

and we can construct solutions for $\rho_0 \in \mathcal{M}(\mathbb{R}^d)$ a nonnegative measure.

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Recall $\delta' \geq \delta^2 - C \sup_{i,j} \|\partial_{ii} u - \partial_{jj} u\|^2_{L^\infty}$.

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➢ **Subcritical mass** $d = 2$. $\mathcal{F}[\rho_0], \mathrm{m}_2[\rho_0] < +\infty$.
Using Log-HLS and conservation of m_2 we get additional integrability.

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta \geq \delta^2(1 - o(1)) \quad \Rightarrow \quad \begin{cases} \delta \geq -C(T)\left(1 + \dfrac{1}{t}\right) \\ \|\rho\|_{L^\infty} \leq C(T)\left(1 + \dfrac{1}{t}\right) \end{cases}$$

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$\sup_{i,j}\|\partial_{ii}u - \partial_{jj}u\|_{L^\infty} \lesssim |\widetilde{\delta}|^{d/(d+2)}\|\rho\|_{L^2}^{4/(d+2)}$ and $\widetilde{\delta} = \max(\delta, \|\rho\|_{L^\infty})$.

➢ **Subcritical mass** $d = 2$. $\mathcal{F}[\rho_0], \mathrm{m}_2[\rho_0] < +\infty$.
Using Log-HLS and conservation of m_2 we get additional integrability.

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta \geq \delta^2(1 - o(1)) \quad \Rightarrow \quad \begin{cases} \delta \geq -C(T)\left(1 + \dfrac{1}{t}\right) \\ \|\rho\|_{L^\infty} \leq C(T)\left(1 + \dfrac{1}{t}\right) \end{cases}$$

➢ **Subcritical mass** $d \geq 3$.

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➢ **Subcritical mass** $d \geq 3$. $\mathcal{F}[\rho_0] < +\infty$. HLS to control $\int \rho^m$.

Li-Yau and Aronson-Bénilan

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➢ **Subcritical mass** $d \geq 3$. $\mathcal{F}[\rho_0] < +\infty$. HLS to control $\int \rho^m$.

➢ **Critical mass**. The result can be extended.

➢ We construct solutions for Keller-Segel.

1. Differential inequality on the pressure.
2. L^∞ estimates.
3. Li-Yau and Aronson-Bénilan estimates.
4. Discussions and new implications.

Discussion

- **Existence of solutions**. Uniform L^{∞} bounds and global in time sol.

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➢ **Smoothing of the initial condition**. For small initial mass the solution is immediately smoothed out. Even if $\rho_0 \in \mathcal{M}(\mathbb{R}^d)$
Is this mass condition a **sharp** threshold???

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Discussion

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Discussion

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➢ L^∞ **bounds**. New **decays** in time for Keller-Segel.

➢ **Lipschitz estimates for non-local aggregation-diffusion PDEs**.

➢ **Liouville and Lane-Emden**. Work in progress.

Thank you for you attention!!!

C. Elbar, A. F-J, and F. Santambrogio. A Li-Yau and Aronson-Bénilan approach for the Keller-Segel with critical exponent. *Preprint in preparation*.

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Δv

$$\partial_t \Delta v = \Delta(\Delta v) + \nabla(\Delta v) \cdot [2\nabla v + \nabla u] + 2|D^2 v|^2 + 2\left(D^2 v - \frac{1}{2}\,\Delta v\,\mathrm{Id}\right) : \left(D^2 u - \frac{1}{2}\,\Delta u\,\mathrm{Id}\right)$$

If $x_0 = x_0(t)$ is a minimum point of $\Delta v(t, \cdot)$

$$\partial_t \Delta v(t, x_0) \geq 2\left|D^2 v(t, x_0)\right|^2 + 2\left(D^2 v(t, x_0) - \frac{1}{2}\,\Delta v(t, x_0)\,\mathrm{Id}\right) : \left(D^2 u(t, x_0) - \frac{1}{2}\,\Delta u(t, x_0)\,\mathrm{Id}\right).$$